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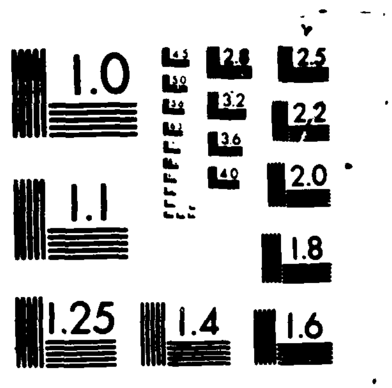

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Professional Paper 455 / April 1987

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Continuous Dependence on Modeling in the Cauchy Problem for Nonlinear Elliptic Equations

by

Allan Bennett

A Division of CNA Hudson Institute

CENTER·FOR·NAVAL·ANALYSES

4401 Ford Avenue • Post Office Box 16268 • Alexandria, Virginia 22302-0268 • (703) 824-2000

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ABSTRACT

The Cauchy problem for various types of second order nonlinear elliptic equations is considered. A substitution $v=\epsilon u$ in the equation leads to a perturbed equation whose solution is compared to an appropriate solution of an unperturbed second order linear elliptic equation obtained by formally setting $\epsilon=0$. In each case a logarithmic convexity argument is used to show that appropriately constrained solutions of the original equation (assumed to exist) are shown to differ from a solution of the associated linear equation in a manner depending continuously on the parameter ϵ.

INTRODUCTION

A problem in ordinary or partial differential equations is said to properly posed if it has a unique solution in the class under consideration and if this solution depends continuously on the data in some appropriate measure. Otherwise the problem is said to be improperly posed. Although Hadamard (8) defined the question of proper posedness at the turn of the century and demonstrated that the Cauchy problem for the Laplace equation is improperly posed, relatively little attention was given to improperly posed problems for partial differential equations until the papers of John (9) and Pucci (16) appeared in the 1950's. Up to that time the prevailing attitude seemed to be that only properly posed problems were of interest in applications.

It is realized now, however, that many problems of physical interest are improperly posed. For example, the Dirichlet problem for a second-order linear elliptic equation on a smooth bounded domain in R^N is properly posed. However, in many physical situations, only a portion of the boundary may be accessible to data measurement. In such cases, one measures additional data--usually the gradient of the unknown function--on that portion of the boundary which is accessible. The resulting problem is an improperly posed Cauchy problem. Payne (14) showed that this problem can be stabilized by imposing an a priori bound on the L^2--norm of the solution. (Special cases of this problem had been considered earlier; for example, Lavrentiev (11) showed that imposing a pointwise bound on the solution of a Cauchy problem for the Laplace equation is sufficient to ensure that the solution will depend continuously upon the Cauchy data in some neighborhood of the data surface.) Extensions of the result in (14) to the Cauchy problem for equations of the type

$$Lu = f(x, u, \text{grad } u) \quad , \tag{1.1}$$

for a uniformly elliptic second-order linear operator L are found in (6, 18-20). In each case a Holder continuous dependence result is obtained by restricting the L^2-norm of the solution.

This work investigates improperly posed Cauchy problems for some second-order nonlinear elliptic equations which cannot be written in the form (1.1) but which can be written as

$$Lv = g(x, v, \text{grad } v, Hv), \tag{1.2}$$

where Hv is the Hessian matrix of v. Examples of such equations are the minimal surface and capillary surface equations. The substitution $v = \epsilon u$ in (1.2) leads to a perturbed equation of the form

$$Lu = \epsilon^k \tilde{g}(x, u, \text{grad } u, Hu) \tag{1.3}$$

for some positive number k. The corresponding unperturbed equation is

$$Lh = 0 \quad , \tag{1.4}$$

and logarithmic convexity arguments are used to derive stability estimates for $v - \epsilon h$, where h is a solution of an appropriate Cauchy problem for (1.4).

The question of the feasibility of approximating a solution of a Cauchy problem for a perturbed equation by a solution of a Cauchy problem for the corresponding unperturbed equation will be referred to as the question of <u>continuous dependence on modeling</u> in the Cauchy problem for the perturbed equation. Adelson's work (1, 2) illustrates this concept. His results apply, for example, to show that an

appropriately constrained solution of the Cauchy problem for the singularly perturbed equation

$$\epsilon\Delta^2 v - v = E(x) \tag{1.5}$$

can be approximated by the solution of a Cauchy problem for

$$-\Delta w = E(x) \tag{1.6}$$

which should, in some sense, be the limiting problem as ϵ tends to zero.

The question of existence of solutions for perturbed problems for all values of ϵ in some interval $(0, \epsilon_0)$ presents no difficulty in most reasonable properly posed problems for ordinary or partial differential equations. Hence, in such problems, one may allow ϵ to tend to zero and prove that the solution of the perturbed equation converges to the solution of the unperturbed equation in some appropriate measure.

However, for improperly posed problems, for given data the solution may fail to exist for some or all values of ϵ in the interval. One can compensate to some extent for this difficulty by allowing for small variations in the data--not an unreasonable thing to do since the data usually cannot be measured exactly. This work shall not be concerned with the complicated questions of existence but shall assume that all solutions under consideration do indeed exist.

2. Notation

Let D be an N-dimensional domain bounded by a closed surface C, and let Σ be that portion of C on which Cauchy data are prescribed. The complement of Σ with respect to C is denoted Σ', and no data are given on Σ'. Assume that the closure $\bar{\Sigma}$ of Σ is a $C^{2+\alpha}$ surface for some $\alpha > 0$. Since Cauchy data are given only on the portion Σ of C, one cannot expect to derive estimates for continuous dependence (on the data) on the entire domain D. In particular one might not expect to derive such estimates for subdomains of D whose boundaries contain a portion of Σ'. Thus a family $\{D_\alpha\}$ of subdomains of D on which to derive stability estimates is chosen as follows:

Let $\{f(x) = \text{constant}\}$ define a set of (not necessarily closed) surfaces. This set is to be chosen so that for each $\alpha \in (0,1]$ the surface $\{f(x) = \alpha\}$ intersects $\bar{D}$ and forms a closed region D_α whose boundary consists only of points on Σ and points on the surface $\{f(x) = \alpha\}$. We set $\Sigma_\alpha = \Sigma \cap \bar{D}_\alpha$ and $S_\alpha = \left\{f(x) = \alpha\right\} \cap \bar{D}_\alpha$.

Assume that $f(x)$ has continuous second derivatives in $\bar{D}_1$. Furthermore, assume that

$$\text{if } \beta \le \gamma \text{, then } D_\beta \le D_\gamma; \tag{2.1}$$

$$|\text{grad } f| \ge \delta > 0 \text{ in } D_1 ; \tag{2.2}$$

$$\Delta f \le 0 \text{ in } D_1 ; \tag{2.3}$$

$$|\Delta f| \le \delta^2 d \text{ in } D_1 ; \tag{2.4}$$

where δ and d are positive constants. (In Section 5, (2.3) and (2.4) are modified somewhat.) Assume that the surfaces have been chosen so that D_α has positive Lebesgue measure for $0 < \alpha \le 1$ but that D_0

has Lebesgue measure zero. For $N \geq 2$, one can choose a raidal harmonic function f which satisfies (2.1) - (2.4).

Throughout this paper commas are used to denote differentiation, and the summation convention is used for repeated indices. For example,

$$u_{,i}\, u_{,i} = \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}\right)^2 = |\nabla u|^2 .$$

The arithmetic-geometric mean (A-G) inequality states that, for positive numbers a, b, and c

$$2ab \leq ca^2 + \frac{1}{c} \cdot b^2 .$$

3. This section examines the minimal surface equation

$$\left[(1 + |\nabla v|^2)^{-1/2} v_{,j}\right]_{,j} = 0 \text{ in } D . \tag{3.1}$$

On Σ , assume that the Cauchy data satisfy

$$\int_\Sigma (v^2 + v_{,i} v_{,i}) \, ds \le \epsilon^2 \tag{3.2}$$

for some small positive number ϵ. The substitution $v = \epsilon u$ in (3.1) yields the perturbed equation

$$\Delta u = \epsilon^2 \rho^2 u_{,i} u_{,ij} \text{ in } D , \tag{3.3}$$

where $\rho = (1 + \epsilon^2 |\nabla u|^2)^{-1/2}$. Formally setting $\epsilon = 0$ in (3.3) gives Laplace's equation

$$\Delta h = 0 \text{ in } D. \tag{3.4}$$

Setting $w = u - h$ yields

$$\Delta w = \epsilon^2 \rho^2 u_{,i} u_{,j} u_{,ij} \text{ in } D . \tag{3.5}$$

Regarding the data on Σ, require, for some $\rho < 6$, that

$$\int_\Sigma (u_{,i} u_{,i})^2 |(u_{,\ell} u_{,j\ell} - u_{,j} u_{,\ell\ell}) n_j| \, ds = 0(\epsilon^{-\rho}) \quad . \tag{3.6}$$

Furthermore assume that

$$\int_{\Sigma} (w^2 + w,_i w,_i) ds = 0(\epsilon^{4-\rho}) \quad . \tag{3.7}$$

It is shown that if u and h are appropriately constrained solutions of (3.3) and (3.4), respectively, which satisfy (3.6) and (3.7), then for $0 < \alpha < \alpha_1 < 1$,

$$\int_{D_\alpha} (v - \epsilon h)^2 dx = 0\left[\epsilon^{(6-\rho)\cdot\nu(\alpha)}\right] \tag{3.8}$$

where ν is a smooth function of α satisfying

$$\nu(0) = 1 \ , \ \nu'(\alpha) < 0, \ \nu(\alpha_1) = 0 \ . \tag{3.9}$$

The following argument closely resembles the one used by Payne (14) when he computed bounds for solutions of ill-posed Cauchy problems for linear elliptic equations. To begin the derivation of (3.8), set, for $\alpha \epsilon [0,1]$,

$$F(\alpha) = Q + \int_0^\alpha (\alpha - \eta) \left\{\int_{D_\eta} [w,_i w,_i + w\Delta w] dx\right\} d\eta \ , \tag{3.10}$$

where Q is given by

$$Q = k_0 \int_{\Sigma} w^2 ds + k_1 \int_{\Sigma} w,_i w,_i ds + k_2 \epsilon^{4-\rho} \quad , \tag{3.11}$$

and the k_i are positive constants which will be chosen later. It is shown that F satisfies a differential inequality of the form

$$FF'' - (F')^2 \geq -C_1FF' - C_2F^2 \tag{3.12}$$

on the interval $(0,\alpha_1)$ for explicit constants C_1 and C_2. The solution of this inequality then leads to the desired bounds.

The first and second derivatives of F are

$$F'(\alpha) = \int_0^\alpha \int_{D_\eta} [w,_i w,_i + w\Delta w] dx d\eta \ , \tag{3.13}$$

$$F''(\alpha) = \int_{D_\alpha} [w,_i w,_i + w\Delta w] dx \ . \tag{3.14}$$

Integrating (3.13) by parts, one can write $F(\alpha)$ and $F'(\alpha)$ in more useful forms:

$$F'(\alpha) = \int_0^\alpha \left\{ \int_{S_\eta} ww,_i f,_i |\nabla f|^{-1} ds + \int_{\Sigma_\eta} ww,_i n,_i ds \right\} d\eta$$

$$- \int_{D_\alpha} ww,_i f,_i dx + \int_0^\alpha \int_{\Sigma_\eta} ww,_i n_i ds d\eta \ . \tag{3.15}$$

Note that on S_η the component n_j of the unit normal is given by $f,_j |\nabla f|^{-1}$. Using (3.15),

$$F(\alpha) = Q + \int_0^\alpha F'(\eta)d\eta = Q + \int_0^\alpha \left\{ \int_{D_\eta} ww,_i f,_i dx + \int_0^\eta \int_{\Sigma_\sigma} ww,_i n_i ds d\sigma \right\} d\eta \; .$$

Integrating by parts above and using (2.3) and the A-G inequality,

$$\begin{aligned} F(\alpha) &= Q + \int_0^\alpha \left\{ \frac{1}{2} \int_{S_\eta} w^2 |\nabla f| ds + \frac{1}{2} \int_{\Sigma_\eta} w^2 f,_i n_i ds \right. \\ &\left. - \frac{1}{2} \int_{D_\eta} w^2 \Delta f dx + \int_0^\eta \int_{\Sigma_\sigma} ww,_i n_i ds d\sigma \right\} d\eta \\ &\geq \frac{1}{2} \int_{D_\alpha} |\nabla f|^2 w^2 dx - \nu_1 \int_\Sigma w^2 ds - \nu_2 \int_\Sigma w,_i w,_i ds + Q \end{aligned} \tag{3.16}$$

for computable constants ν_1 and ν_2 . One can now choose the constants k_i in Q so that

$$\frac{1}{2}\left[\int_{D_\alpha} rw^2 dx + Q \right] \leq F(\alpha) \leq \frac{d+1}{2} \left[\int_{D_\alpha} rw^2 dx + Q \right] \; . \tag{3.17}$$

where $r = |\nabla f|^2$.

The inequality (3.17) enables the use of the solution of (3.12) to estimate the L^2-integrals of w over the domains D_α. To derive (3.12), we need three preliminary estimates.

Lemma 1: Let $\alpha_1 \epsilon (0,1)$. Then for $\alpha \epsilon (0,\alpha_1)$

$$\int_{D_\alpha} (\Delta w)^2 dx \le c\epsilon^4 \left\{ \int_{D_1} [(u,_i u,_i)^3 (1 + \epsilon^2 u,_j u,_j)] dx \right.$$

$$\left. + \int_{\Sigma_1} (u,_i u,_i)^2 | (u,_\ell u,_{j\ell} - u,_j u,_{\ell\ell}) n_j | ds \right\}$$

for an explicit constant c independent of ϵ and α.

Proof: Define the function ω on D_1 by

$$\omega(X) = \begin{cases} 1 & \text{in } D_\alpha \cup \Sigma_\alpha \\ \dfrac{1-f(x)}{1-\alpha} & \text{in } \bar{D}_1 - (D_\alpha \cup \Sigma_\alpha) \, . \end{cases}$$

Note that $\omega = 0$ on S_1 and that $|\omega| \le 1$ in D_1. Since $f \epsilon C^2(\bar{D}_1)$, the first and second derivatives of ω are uniformly bounded on D_1.

Now consider

$$J(\alpha) = \epsilon^{-4} \int_{D_\alpha} (\Delta w)^2 dx \le \int_{D_1} \omega^2 \rho^4 (u,_i u,_i)^2 u,_{jk} u,_{jk} dx$$

$$\leq \int_{D_1} \omega^2 \rho^2 [(u,_i u,_i)^2 u,_{jk} u,_{jk} \tag{3.18}$$

$$+ 4(u,_i u,_i) u,_k u,_\ell u,_{jk} u,_{j\ell}] dx \; .$$

The second inequality above holds since $u,_k u,_\ell u,_{jk} u,_{j\ell} \geq 0$.

By Schwarz's inequality and the A-G inequality,

$$(u,_j u,_\ell u,_{j\ell})^2 \leq |\nabla u|^2 u,_k u,_\ell u,_{jk} u,_{j\ell} \leq |\nabla u|^4 u,_{j\ell} u,_{j\ell} \; .$$

Therefore, from (3.18),

$$J(\alpha) \leq \int_{D_1} \omega^2 \rho^4 (u,_i u,_i)^2 [u,_{j\ell} u,_{j\ell} (1+\epsilon^2 |\nabla u|^2)^2 - \epsilon^4 (u,_j u,_\ell u,_{j\ell})^2] dx$$

$$+ 4 \int_{D_1} \omega^2 \rho^2 (u,_i u,_i) [u,_k u,_\ell u,_{jk} u,_{j\ell} (1+\epsilon^2 |\nabla u|^2) - \epsilon^2 (u,_j u,_\ell u,_{j\ell})^2] dx$$

$$= \int_{D_1} \omega^2 (u,_i u,_i)^2 \left\{ u,_{j\ell} u,_{j\ell} - (\Delta u)^2 \right\} dx$$

$$+ 4 \int_{D_1} \omega^2 (u,_i u,_i) \left\{ u,_k u,_\ell u,_{jk} u,_{j\ell} - \Delta u (u,_j u,_\ell u,_{j\ell}) \right\} dx \; .$$

To eliminate the second-order terms appearing in the bound for $J(\alpha)$, integrate the first term on the right side of (3.19) by parts. First note that

$$\int_{D_1} \omega^2(u,_i u,_i)^2 u,_{j\ell} u,_{j\ell} dx = \int_{\Sigma_i} \omega^2(u,_i u,_i)^2 u,_j u,_{j\ell} n_\ell ds$$

$$- \int_{D_1} \omega^2(u,_i u,_i)^2 u,_j (\Delta u),_j dx - 2\int_{D_1} \omega\omega,_\ell (u,_i u,_i)^2 u,_j u,_{j\ell} dx$$

$$- 4\int_{D_1} \omega^2(u,_i u,_i)(u,_j u,_k u,_{j\ell} u,_{k\ell}) dx . \tag{3.20}$$

Integrate the first volume integral on the right side of (3.20) by parts to obtain

$$- \int_{D_1} \omega^2(u,_i u,_i)^2 u,_j (\Delta u),_j dx = - \int_{\Sigma_1} \omega^2(u,_i u,_i)^2 u,_j u,_{\ell\ell} n_j ds$$

$$+ \int_{D_1} \omega^2(\Delta u)^2(u,_i u,_i)^2 dx + 2\int_{D_1} \omega\omega,_j (u,_i u,_i)^2 u,_j u,_{\ell\ell} dx$$

$$+ 4\int_{D_1} \omega^2(u,_i u,_i)^2 u,_j u,_k u,_{jk} u,_{\ell\ell} dx . \tag{3.21}$$

Combining (3.20) and (3.21) yields

$$\int_{D_1} \omega^2(u,_i u,_i)^2 u,_{j\ell} u,_{j\ell} dx = \int_{\Sigma_1} \omega^2(u,_i u,_i)^2 [u,_j u,_{j\ell} n_\ell - u,_j u,_{\ell\ell} n_j] ds$$

$$+ \int_{D_1} \omega^2(\Delta u)^2 (u,_i u,_i)^2 dx + 2\int_{D_1} \omega\omega,_j (u,_i u,_i)^2 u,_{\ell\ell} u,_j dx$$

$$+ 4\int_{D_1} \omega^2(u,_i u,_i) u,_j u,_k u,_{jk} u,_{\ell\ell} dx - 2\int_{D_1} \omega\omega,_\ell (u,_i u,_i)^2 u,_j u,_{j\ell} dx$$

$$- 4\int_{D_1} \omega^2(u,_i u,_i)(u,_j u,_k u,_{jk}) dx \,. \tag{3.22}$$

Returning to (3.19),

$$\int_{D_1} \omega^2 \rho^4 (u,_i u,_i)^2 [u,_{j\ell} u,_{j\ell} (1+\epsilon^2|\nabla u|^2)^2 - \epsilon^4 (u,_j u,_\ell u,_{j\ell})^2] dx$$

$$+ 4\int_{D_1} \omega^2 \rho^2 (u,_i u,_i)[u,_k u,_\ell u,_{k\ell}(1+\epsilon^2|\nabla u|^2) - \epsilon^2 (u,_j u,_\ell u,_{j\ell})^2] dx$$

$$= \int_{\Sigma_1} \omega^2 (u,_i u,_i)^2 (u,_j u,_{j\ell} n_\ell - u,_j u,_{\ell\ell} n_j) ds$$

$$+ 2\int_{D_1} \omega(u,_i u,_i)^2(\omega,_j u,_j u,_{\ell\ell} - \omega,_\ell u,_j u,_{j\ell})dx \; . \tag{3.23}$$

Using the A-G inequality, the absolute value of the volume integral on the right side of (3.23) is bounded above by

$$k\int_{D_1} (u,_i u,_i)^3(1+\epsilon^2|\nabla u|^2)dx + \frac{1}{2}\int_{D_1} \omega^2\rho^2(u,_i u,_i)^2 u,_{j\ell}u,_{j\ell}dx \tag{3.24}$$

for a computable constant k. Thus from (3.18), (3.19), (3.23), and (3.24),

$$\frac{1}{2}J(\alpha) \le \frac{1}{2}\int_{D_1} \omega^2\rho^2[(u,_i u,_i)^2 u,_{jk}u,_{jk} + 4(u,_i u,_i)u,_k u,_\ell u,_{jk}u,_{j\ell}]dx$$

$$\le \frac{1}{2}\int_{D_1} \omega^2\rho^2(u,_i u,_i)^2 u,_{jk}u,_{jk}dx$$

$$+ 4\int_{D_1} \omega^2\rho^2(u,_i u,_i)u,_k u,_\ell u,_{jk}u,_{j\ell}dx$$

$$\le \int_{\Sigma_1} \omega^2(u,_i u,_i)^2[u,_j u,_{j\ell}n_\ell - u,_j u,_{\ell\ell}n_j]ds$$

$$+ k \int_{D_1} (u_{,i}u_{,i})^3(1 + \epsilon^2|\nabla u|^2)dx \ .$$

The proof of Lemma 1 is now complete.

Before deriving the next two required estimates, it is convenient at this point to place a constraint on the function u. Using the result of Lemma 1 and the definition of Q as a guide, require that

$$\int_{D_1} [(u_{,i}u_{,i})^3(1 + \epsilon^2 u_{,j}u_{,j})]dx = O(\epsilon^{-\rho}) \ . \tag{3.25}$$

This constraint is used in the proofs of the next two lemmas, which are understood to be valid on the interval $(0, \alpha_1)$.

<u>Lemma 2: If $F(\alpha)$ is given by (3.10), then</u>

$$|F'| \le F' + K_1 F \tag{3.26}$$

<u>for an explicit constant K_1.</u>

<u>Proof</u>: From (3.13) it is immediate that

$$|F'| \le F' + 2\left| \int_0^\alpha \int_{D_\eta} w\Delta w dx d\eta \right| \ . \tag{3.27}$$

By the A-G inequality,

$$2\left|\int_0^\alpha\int_{D_\eta} w\Delta w dx d\eta\right| \le \int_0^\alpha\int_{D_\eta} w^2 dx d\eta + \int_0^\alpha\int_{D_\eta} (\Delta w)^2 dx d\eta ,$$

so that (3.17) and Lemma 1 yield

$$|F'(\alpha)| \le F'(\alpha) + 2\delta^{-2}F(\alpha) + 0(\epsilon^{4-p}) .$$

This completes the proof of Lemma 2.

Lemma 3: If $F(\alpha)$ is given by (3.10), then

$$\int_{D_\alpha} [w,_i w,_i - 2r^{-1}(w,_i f,_i)^2]dx \ge -K_2F' - K_3F , \tag{3.28}$$

for explicit constants K_2 and K_3.

Proof: To establish this lemma, consider the identity (also used in Payne (14))

$$2\int_{D_\alpha} (\alpha-\eta)r^{-1}f,_k w,_k \Delta,_k \Delta w dx = \int_0^\alpha\oint_{\partial D_\eta} \left[2f,_k w,_k w,_i n_i - f,_i n_i |\nabla w|^2\right] r^{-1} ds d\eta$$

$$- \int_{D_\alpha} (\alpha-\eta)\left[2(f,_\ell r^{-1}),_j w,_j w,_\ell - (f,_\ell r^{-1}),_\ell |\nabla w|^2\right] dx . \tag{3.29}$$

The integrals over S_η may be rewritten as

$$\int_0^\alpha \int_{S_\eta} \left[2f_{,k}w_{,k}w_{,i}n_i - f_{,i}n_i|\nabla w|^2\right] r^{-1} ds d\eta$$

$$= \int_{D_\alpha} \left[2(f_{,i}w_{,i})^2 r^{-1} - |\nabla w|^2\right] dx \; . \tag{3.30}$$

Substituting (3.30) into (3.29) and using the A-G inequality,

$$\int_{D_\alpha} \left[w_{,i}w_{,i} - 2r^{-1}(w_{,i}f_{,i})^2\right] dx \geq - k_3 \int_\Sigma w_{,i}w_{,i} ds$$

$$- k_4 \int_{D_\alpha} (\alpha-\eta) w_{,i}w_{,i} dx - k_5 \int_{D_\alpha} (\alpha-\eta) f_{,k}w_{,k} \Delta w dx \; . \tag{3.31}$$

(The k_i in this proof are all explicit positive constants.) Clearly

$$- k_3 \int_\Sigma w_{,i}w_{,i} ds \geq - k_6 F \; . \tag{3.32}$$

Note that

$$\int_{D_\alpha} (\alpha-\eta) w_{,i}w_{,i} dx = \int_0^\alpha \int_{D_\eta} w_{,i}w_{,i} dx d\eta \; , \tag{3.33}$$

and the proofs of Lemmas 1 and 2 yield the conclusion that

$$- k_4 \int_{D_\alpha} (\alpha-\eta) w_{,i} w_{,i} dx \geq - k_7 F' - k_8 F \; . \tag{3.34}$$

Finally,

$$\left| \int_{D_\alpha} (\alpha-\eta) f_{,k} w_{,k} \Delta w dx d\eta \right| = \left| \int_0^\alpha \int_{D_\eta} f_{,k} w_{,k} \Delta w dx d\eta \right|$$

$$\leq k_9 \int_0^\alpha \int_{D_\eta} w_{,k} w_{,k} dx d\eta + k_{10} \int_0^\alpha \int_{D_\eta} (\Delta w)^2 dx d\eta$$

$$\leq k_{11} F' + k_{12} F \; . \tag{3.35}$$

Combining (3.32), (3.34), and (3.35), one sees that Lemma 3 holds.

We can now proceed to derive the inequality (3.12). Using (3.17),

$$FF'' - (F')^2 \geq \left\{ \left[\frac{1}{2} \int_{D_\alpha} r w^2 dx \right] \left[\int_{D_\alpha} w_{,i} w_{,i} dx \right] - \left(\int_{D_\alpha} w w_{,i} f_{,i} dx \right)^2 \right\}$$

$$+ F \int_{D_\alpha} w \Delta w dx - 2|F'| \cdot \left| \int_0^\alpha \int_{\Sigma_\eta} w w_{,i} n_i ds d\eta \right| \; . \tag{3.36}$$

In arriving at (3.36), we have dropped a number of nonnegative terms on the right. On account of (3.25), one can find a constant k_{12} so that

$$\left| \int_{D_\alpha} w\Delta w dx \right| \le k_{12}F , \tag{3.37}$$

and the A-G inequality gives a computable constant k_{13} such that

$$\left| \int_0^\alpha \int_{\Sigma_\eta} ww,_i n_i ds d\eta \right| \le k_{13}F . \tag{3.38}$$

For the term in braces in (3.36), by the Schwarz inequality, Lemma 3, (3.17), and Lemma 2,

$$\int_{D_\alpha} rw^2 dx \int_{D_\alpha} w,_i w,_i dx - 2\left(\int_{D_\alpha} ww,_i f,_i dx \right)^2$$

$$\ge \int_{D_\alpha} rw^2 dx \left\{ \int_{D_\alpha} w,_i w,_i dx - 2 \int_{D_\alpha} r^{-1}(w,_i f,_i)^2 dx \right\}$$

$$\ge - 2K_3 FF' - 2(K_1K_3 + K_4)F^2 . \tag{3.39}$$

Applying Lemma 2 now to the term $|F'|$ in (3.36) gives (3.12) for explicit constants C_1 and C_2.

It is well-known (see, e.g., Levine (12)) that a solution F of (3.12) which vanishes for one value of α in the interval $[0,\alpha_1]$ must vanish identically. Thus, assume without loss that $F(\alpha) > 0$ for all α in $[0, \alpha_1]$. Set

$$\sigma = \exp(-c_1\alpha), \; G(\sigma) = \log F(\alpha) \cdot \sigma^{-C_2/C_1^2} , \tag{3.40}$$

to see that

$$G''(\sigma) = (C_1F\sigma)^{-2}[C_2F^2 + FF'' + C_1FF' - (F')^2] \geq 0 . \tag{3.41}$$

Hence G is a convex function of σ, so that by Jensen's inequality,

$$F(\alpha)\sigma^{-C_2/C_1^2} \leq \left[F(\alpha_1)\sigma_1^{-C_2/C_1^2}\right]^{\frac{1-\sigma}{1-\sigma_1}}\left[F(0)\right]^{\frac{\sigma-\sigma_1}{1-\sigma_1}} , \tag{3.42}$$

where

$$\sigma_1 = \exp(-C_1\alpha_1) . \tag{3.43}$$

Note that $F(0) = Q$, which is $0(\epsilon^{4-p})$ by assumption.

As has been noted in earlier papers (John (10), Pucci (17)), in order to make $F(\alpha)$ small for $0 < \alpha < \alpha_1$, it does not suffice to make $F(0)$ small. One must ensure that $F(\alpha_1)$ is not excessively large. On account of (3.17), it is necessary to constrain the L^2-norms of u and h on D_1. Thus, assume that

$$\int_{D_1} u^2 dx = O(\epsilon^{-2}), \quad \int_{D_1} h^2 dx = O(\epsilon^{-2}) . \tag{3.44}$$

One can then compute a constant N_1 independent of ϵ so that

$$F(\alpha_1)\sigma_1^{-C_2/C_1^2} \le N_1^2 \epsilon^{-2} . \tag{3.45}$$

Insertion of (3.45) into (3.42) gives

$$F(\alpha) = O\left(\epsilon^{[2(\sigma-1)+(4-p)(\sigma-\sigma_1)]/1-\sigma_1}\right) . \tag{3.46}$$

From (3.17), we now obtain the following.

<u>Theorem 1: If u and h are solutions of (3.3) and (3.4) respectively, which satisfy the boundary conditions (3.6) and (3.7) as well as the constraints (3.25) and (3.44), then the solution v of (3.1) satisfies the continuous dependence inequality</u>

$$\int_{D_\alpha} (v - \epsilon h)^2 dx = O\left(\epsilon^{(6-p)\nu(\alpha)}\right) \qquad \underline{\text{for } 0 \le \alpha \le \alpha_1}$$

<u>where</u> $\nu(\alpha) = (\sigma - \sigma_1)/(1 - \sigma_1)$.

We close this section with some additional remarks. The reason for deriving (3.12) only on $(0, \alpha_1)$ with $\alpha_1 < 1$ is that the derivatives of the function w in the proof of Lemma 1 become unbounded as α approaches 1. By restricting attention to an interval $(0, \alpha_2)$ with $\alpha_2 < \alpha_1$, one can derive bounds for the Dirichlet integral of $v-\epsilon h$ as follows:

$$\text{Let } \mu(x) = \begin{cases} 1 & \text{in } D_\beta \cup \Sigma_\beta \\ \dfrac{\alpha_3 - f(x)}{\alpha_3 - \beta} & \text{in } D_{\alpha_3} - (D_\beta \cup \Sigma_\beta) \end{cases}.$$

where $\beta < \alpha_2$ and α_3 is a fixed number between α_2 and α_1. Then

$$\int_{D_\beta} (u - h)_{,i}(u - h)_{,i}\, dx \le \int_{D_{\alpha_3}} \mu^2 (u - h)_{,i}(u - h)_{,i} dx$$

$$= \int_{\Sigma_{\alpha_3}} \mu^2 (u - h)(u - h)_{,i} n_i ds - \int_{D_{\alpha_3}} (u - h)(u - h)_{,i} (\mu^2)_{,i} dx$$

$$- \int_{D_{\alpha_3}} \mu^2 (u - h)(u - h)_{,ii} dx$$

$$= 0(\epsilon^{4-p}) + \frac{1}{2} \int_{D_{\alpha_3}} (u - h)^2 \Delta(\mu^2) dx - \int_{D_{\alpha_3}} \mu^2 (u - h)\Delta(u - h) dx . \qquad (3.47)$$

Using Theorem 1, one can show that the first volume integral on the right side of (3.47) is $0\left(\epsilon^{(6-p)\cdot\nu(\alpha_3)-2}\right)$. Using the A-G inequality, Lemma 1, and Theorem 1, we can show that the second volume integral on the right side of (3.47) is also $0\left(\epsilon^{6-p)\cdot\nu(\alpha_3)-2}\right)$. Thus,

$$\int_{D_\beta} |\nabla(v - \epsilon h)|^2 dx = 0\left(\epsilon^{(6-p)\nu(\alpha_3)}\right)$$

for $\beta < \alpha_2$.

4. In this section, we consider the capillary surface equation

$$((1 + |\nabla v|^2)^{-1/2} v,_j),_j = cv \quad \text{in} \quad D \ , \tag{4.1}$$

where c is a positive constant. On the surface Σ, assume that the Cauchy data satisfy

$$\int_{\Sigma} (v^2 + |\nabla v|^2) ds \le \epsilon^2 \tag{4.2}$$

for some small positive number ϵ. The substitution $v = \epsilon u$ in (4.1) yields the perturbed equation

$$\Delta u - cu = \epsilon^2 \rho^2 u,_i u,_j u,_{ij} + c\left(\frac{1}{\rho} - 1\right)u \tag{4.3}$$

where

$$\rho = (1 + \epsilon^2 |\nabla u|^2)^{-1/2} \ .$$

Since

$$\frac{1}{\rho} - 1 = O(\epsilon^2)|\nabla u|^2 \ ,$$

compare u to a function h satisfying

$$\Delta h = ch \quad \text{in} \quad D \ . \tag{4.4}$$

Setting $w = u - h$,

$$\Delta w - cw = \epsilon^2 \rho^2 u,_i u,_j u,_{ij} + \left(\frac{1}{\rho} - 1\right) cu \ . \tag{4.5}$$

Under appropriate constraints on u and h, we obtain a continuous dependence inequality for $v - \epsilon h = \epsilon(u - h)$ which is similar to that of the previous section. In order to choose appropriate constraints as

well as to prove analogs of Lemmas 2 and 3 of Section 3, we need the following analog of Lemma 1.

Lemma 4: Let $\alpha_1 \epsilon (0,1)$. Then there are constants ℓ_1 and ℓ_2 independent of ϵ so that for each $\alpha \epsilon (0,1)$

$$\int_{D_\alpha} (\Delta w)^2 dx \le \ell_1 \int_{D_\alpha} w^2 dx$$

$$+ \ell_2 \epsilon^4 \left\{ \int_\Sigma (u,_i u,_i)^2 | u,_j u,_{j\ell} n_\ell - u,_j u,_{\ell\ell} n_j | ds \right.$$

$$+ \int_{D_1} \left[(u,_i u,_i)^3 (1 + \epsilon^2 |\nabla u|^2) + u^2 (u,_i u,_i)^2 (1 + \epsilon^2 |\nabla u|^2)^2 \right.$$

$$\left. \left. + |u| \rho^{-1} (u,_i u,_i) \right] dx \right\} .$$

Proof: From (4.5) and the A-G inequality

$$(\Delta w)^2 \le c_1 w^2 + c_2 \epsilon^4 u^2 |\nabla u|^4 + c_3 \epsilon^4 \rho^4 (u,_i u,_j u,_{ij})^2 \tag{4.6}$$

for computable constants c_1, c_2, and c_3. As in section 3, consider

$$J(\alpha) = \int_{D_\alpha} \rho^4 (u,_i u,_j u,_{ij})^2 dx \tag{4.7}$$

and note that

$$
\begin{aligned}
J(\alpha) &\le \int_{D_\alpha} \omega^2 \rho^4 (u_{,i} u_{,i})^2 u_{,jk} u_{,jk} dx \\
&\le \int_{D_1} \omega^2 \rho^2 \left[(u_{,u} u_{,i})^2 u_{,jk} u_{,jk} + 4 (u_{,i} u_{,i}) u_{,k} u_{,\ell} u_{,jk} u_{,j\ell} \right] dx \\
&\le \int_{D_1} \omega^2 \rho^4 (u_{,i} u_{,i})^2 \left[u_{,j\ell} u_{,j\ell} (1 + \epsilon^2 |\nabla u|^2)^2 - \epsilon^4 (u_{,j} u_{,\ell} u_{,j\ell})^2 \right] dx \\
&\quad + 4 \int_{D_1} \omega^2 \rho^2 (u_{,i} u_{,i}) \left[(u_{,k} u_{,\ell} u_{,jk} u_{,j\ell})(1 + \epsilon^2 |\nabla u|^2) - \epsilon^2 (u_{,j} u_{,\ell} u_{,j\ell})^2 \right] dx \\
&\equiv A \,,
\end{aligned}
\tag{4.8}
$$

where $\omega(x)$ is defined as in section 3. As in the previous section, note that

$$
\begin{aligned}
\int_{D_1} \omega^2 (u_{,i} u_{,i})^2 u_{,j\ell} u_{,j\ell} dx &= \int_{\Sigma_1} \omega^2 (u_{,i} u_{,i})^2 [u_{,j} u_{,j\ell} n_\ell - u_{,j} u_{,\ell\ell} n_j] ds \\
&\quad + \int_{D_1} \omega^2 (\Delta u)^2 (u_{,i} u_{,i})^2 dx + 2 \int_{D_1} \omega (u_{,i} u_{,i})^2 [\omega_{,j} u_{,\ell\ell} u_{,j} - \omega_{,\ell} u_{,j} u_{,j\ell}] dx
\end{aligned}
$$

$$+ 4 \int_{D_1} \omega^2 (u,_i u,_i) u,_j u,_k u,_{jk} u,_{\ell\ell} dx$$

$$- 4 \int_{D_1} \omega^2 (u,_i u,_i)(u,_j u,_k u,_{j\ell} u,_{k\ell}) dx \ . \qquad (4.9)$$

Use the equation (4.3) to write

$$- \int_{D_1} \omega^2 (u,_i u,_i)^2 (\epsilon^2 \rho^2 u,_j u,_\ell u,_{j\ell})^2 dx$$

$$= - \int_{D_1} \omega^2 (u,_i u,_i)^2 \left[\Delta u - c \frac{1}{\rho} u\right]^2 dx$$

$$= - \int_{D_1} \omega^2 (u,_i u,_i)^2 (\Delta u) dx - c^2 \int_{D_1} \omega^2 (u,_i u,_i)^2 \rho^{-2} u^2 dx$$

$$+ 2c \int_{D_1} \omega^2 (u,_i u,_i)^2 u \Delta u \rho^{-1} dx \ . \qquad (4.10)$$

Combining (4.9) and (4.10),

$$\int_{D_1} \omega^2 (u,_i u,_i)^2 \rho^4 \left[u,_{j\ell} u,_{j\ell} (1 + \epsilon^2 |\nabla u|^2)^2 - \epsilon^4 (u,_j u,_\ell u,_{j\ell})^2\right] dx$$

$$- \int_{\Sigma_1} \omega^2 (u,_i u,_i)^2 [u,_j u,_{j\ell} n_\ell - u,_j u,_{\ell\ell} n_j] ds$$

$$+ 2 \int_{D_1} \omega (u,_i u,_i)^2 [\omega,_j u,_{\ell\ell} u,_j - \omega,_\ell u,_j u,_{j\ell}] dx$$

$$+ 4 \int_{D_1} \omega^2 (u,_i u,_i) u,_j u,_k u,_{jk} u,_{\ell\ell} dx - 4 \int_{D_1} \omega^2 (u,_i u,_i)(u,_j u,_k u,_{j\ell} u,_{k\ell}) dx$$

$$- c^2 \int_{D_1} \omega^2 (u,_i u,_i)^2 \rho^{-2} u^2 dx + 2c \int_{D_1} \omega^2 (u,_i u,_i)^2 u (\Delta u) \rho^{-1} dx \; . \tag{4.11}$$

Returning to (4.8), we use (4.3) to write

$$4 \int_{D_1} \omega^2 \rho^2 (u,_i u,_i)[u,_k u,_\ell u,_{jk} u,_{j\ell} \rho^{-2} - \epsilon^2 (u,_j u,_\ell u,_{j\ell})^2] dx$$

$$= 4 \int_{D_1} \omega^2 (u,_i u,_i)(u,_k u,_\ell u,_{jk} u,_{j\ell}) dx$$

$$- 4 \int_{D_1} \omega^2 (u,_i u,_i)(u,_j u,_\ell u,_{j\ell}) \left[\Delta u - c \frac{1}{\rho} u \right] dx \; . \tag{4.12}$$

Adding (4.11) to (4.12),

$$A = \int_{\Sigma_1} \omega^2 (u_{,i}u_{,i})^2 [u_{,j}u_{,j\ell}n_\ell - u_{,j}u_{,\ell\ell}n_j] ds$$

$$+ 2 \int_{D_1} \omega (u_{,i}u_{,i})^2 [\omega_{,j}u_{,\ell\ell}u_{,j} - \omega_{,\ell}u_{,j}u_{,j\ell}] dx$$

$$- c^2 \int_{D_1} \omega^2 (u_{,i}u_{,i})^2 \rho^{-2} u^2 dx$$

$$+ 2c \int_{D_1} \omega^2 (u_{,i}u_{,i})^2 u(\Delta u)\rho^{-1} dx$$

$$+ 4c \int_{D_1} \omega^2 (u_{,i}u_{,i})\rho^{-1} u \, dx \ . \qquad (4.13)$$

Using the A-G inequality, one can find a constant k so that

$$A \le \int_{\Sigma_1} \omega^2 (u_{,i}u_{,i})^2 [u_{,j}u_{,j\ell}n_\ell - u_{,j}u_{,\ell\ell}n_j] ds$$

$$+ k \int_{D_1} (u_{,i}u_{,i})^3 (1 + \epsilon^2 |\nabla u|^2) dx$$

$$+ \frac{1}{2} \int_{D_1} \omega^2 \rho^2 (u_{,i} u_{,i})^2 u_{,j\ell} u_{,j\ell} dx$$

$$+ k \int_{D_1} \omega^2 (u_{,i} u_{,i})^2 u^2 (1 + \epsilon^2 |\nabla u|^2)^2 dx$$

$$+ 4c \int_{D_1} \omega^2 (u_{,i} u_{,i}) \rho^{-1} u \, dx \; . \tag{4.14}$$

We can now conclude that

$$J(\alpha) \le \ell \Bigg\{ \int_{\Sigma_1} \omega^2 (u_{,i} u_{,i})^2 (u_{,j} u_{,j\ell} n_\ell - u_{,j} u_{,\ell\ell} n_j) ds$$

$$+ \int_{D_1} (u_{,i} u_{,i})^3 (1 + \epsilon^2 |\nabla u|^2) dx$$

$$+ \int_{D_1} u^2 (u_{,i} u_{,i})^2 (1 + \epsilon^2 |\nabla u|^2)^2 dx$$

$$+ \int_{D_1} |u| \rho^{-1} (u_{,i} u_{,i}) dx \Bigg\} \; , \tag{4.15}$$

for an explicit constant ℓ . Thus Lemma 4 holds.

Having obtained an estimate for

$$\int_{D_\alpha} (\Delta w)^2 dx ,$$

one can argue in a manner similar to that of the preceding section. For $\alpha \epsilon [0,1]$ set

$$F(\alpha) = Q + \int_0^\alpha (\alpha-\eta)\left\{ \int_{D_\eta} [w,_i w,_i + w\Delta w] dx \right\} d\eta \tag{4.16}$$

where Q is given by

$$Q = k_0 \int_\Sigma w^2 ds + k_1 \int_\Sigma w,_i w,_i ds + k_2 \epsilon^{4-p} \tag{4.17}$$

with $2 \leq p < 6$. The k_i are positive constants chosen so that (3.17) holds. Assume that

$$\int_\Sigma (w^2 + w,_i w,_i) ds = 0(\epsilon^{4-p}) , \tag{4.18}$$

and also impose the constraint

$$\int_{\Sigma} (u,_i u,_i)^2 |u,_j u,_{j\ell} n_\ell - u,_j u,_{\ell\ell} n_j| ds$$

$$+ \int_{D_1} [(u,_i u,_i)^3 (1 + \epsilon^2 |\nabla u|^2) + u^2 (u,_i u,_i)^2 (1 + \epsilon^2 |\nabla u|^2)^2$$

$$+ |u| \rho^{-1} (u,_i u,_i)] dx = O(\epsilon^{-p}) . \qquad (4.19)$$

We now state the remaining estimates, which are understood to hold on the interval $(0, \alpha_1)$. The proofs are similar to those of the previous section.

Lemma 5: If $F(\alpha)$ is given by (4.16), then

$$|F'| \le F' + K_1 F \qquad (4.20)$$

for a computable constant K_1.

Lemma 6: If $F(\alpha)$ is given by (4.16), then

$$\int_{D_\alpha} [w,_i w,_i - 2r^{-1} (w,_i f,_i)^2] dx \ge - K_2 F' - K_3 F , \qquad (4.21)$$

for explicit constants K_2 and K_3. (recall that $r = |\nabla f|^2$.)

One may now conclude as in the last section that on the interval $(0, \alpha_1)$,

$$FF'' - (F')^2 \geq - C_1FF' - C_2F^2$$

for explicit constants C_1 and C_2. Assuming that

$$\int_{D_1} u^2dx = O(\epsilon^{-2}) \ , \quad \int_{D_1} h^2dx = O(\epsilon^{-2}) \ , \tag{4.22}$$

we have

Theorem 2: If u and h are solutions of (4.3) and (4.4), respectively, which satisfy (4.18), (4.19), and (4.22), then for $0 < \alpha < \alpha_1$

$$\int_{D_\alpha} (v - \epsilon h)^2dx = O(\epsilon^{\gamma(\alpha)})$$

with $\gamma(\alpha) = (6 - p)(\sigma - \sigma_1)/(1 - \sigma_1)$, $\sigma = \exp(-C_1\alpha)$, and $\sigma_1 = \exp(-C_1\alpha_1)$.

As in section 3, one can find a continuous dependence inequality for the Dirichlet integral of $v - \epsilon h$. Introduce the function μ as in the last section with $\beta < \alpha_2$ and a fixed number α_3 between α_2 and α_1. Then

$$\int_{D_\beta} (u-h)_{,i}(u-h)_{,i}dx \leq \int_{D_{\alpha_3}} \mu^2(u-h)_{,i}(u-h)_{,i}dx$$

$$= \int_{\Sigma_{\alpha_3}} \mu^2(u-h)(u-h)_{,i}n_i ds - \int_{D_{\alpha_3}} (u-h)(u-h)_{,i}(\mu^2)_{,i}dx$$

$$- \int_{D_{\alpha_3}} \mu^2(u-h)\Delta(u-h)dx$$

$$= 0(\epsilon^{4-p}) + \frac{1}{2}\int_{D_{\alpha_3}} (u-h)^2\Delta(\mu^2)dx$$

$$- \int_{D_{\alpha_3}} \mu^2(u-h)\Delta(u-h)dx \ . \qquad (4.23)$$

Using the A-G inequality and the estimates of this section, we conclude that

$$\int_{D_\beta} (v - \epsilon h)_{,i}(v - \epsilon h)_{,i}dx = 0\left(\epsilon^{\gamma(\alpha_3)}\right)$$

with $\gamma(\alpha)$ given as in Theorem 2.

5. In this section, we examine a general second-order nonlinear elliptic equation of the form

$$(a^{ij}(x,\ v(x))v,_i),_j = g(x,\ v,\ \text{grad } v) \quad \text{in} \quad D\ . \tag{5.1}$$

Assume that $g(x,\ p,\ \xi) = O(|p| + |\xi|)$ for $x \in \bar{D}$, $p \in R$, and $\xi \in R^N$. Also assume that the a^{ij} are C^1 functions in the $\bar{D} \times R$ with $a^{ij} = A^{ji}$ and that for $1 \le i,j,k \le N$

$$\begin{array}{l}\text{the functions } a^{ij}(x,p) \text{ and their first derivatives}\\ \text{with respect to the variables } x_k \text{ are uniformly}\\ \text{Lipschitz continuous in } \bar{D} \times R\ .\end{array} \tag{5.2}$$

To ensure that equation (5.1) is uniformly elliptic, we require that there be positive constants a_0 and a_1 so that

$$a_0|\xi|^2 \le a^{ij}(x,p)\xi_i\xi_j \le a_1|\xi|^2 \tag{5.3}$$

for each $x \in \bar{D}$, $p \in R$, and $\xi \in R^N$. We use the notation

$$b^{ij}(x) = a^{ij}(x,v(x)) \tag{5.4}$$

for a fixed solution v of (5.1) to be considered here. Thus,

$$(b^{ij}),_k(x) = (a^{ij}),_k(x,v(x)) + (a^{ij}),_p(x,v(x))v,_k\ . \tag{5.5}$$

Consider the Cauchy problem for (5.1) with v and grad v measured on Σ. The continuous dependence estimate derived for v is less sharp than those of the previous sections.

As in the preceding sections, assume that

$$\int_{\Sigma} (v^2 + v_{,i} v_{,i}) ds \le \epsilon^2 \tag{5.6}$$

for some small positive number ϵ. Making the formal substitution $v = \epsilon u$ leads to the following perturbed equation for a function u:

$$a^{ij}(x,\epsilon u)u_{,ij} + (a^{ij})_{,j}(x,\epsilon u)u_{,i} + \epsilon(a^{ij})_{,p}(x,\epsilon u)u_{,i}u_{,j}$$

$$= O(\epsilon)(|u| + |\nabla u|) \; . \tag{5.7}$$

Compare u to a solution h of the linear equation

$$(A^{ij}(x)h_{,i})_{,j} = 0 \tag{5.8}$$

where $A^{ij}(x) = a^{ij}(x, 0)$. Subtracting (5.8) from (5.7),

$$(A^{ij}(x)w_{,i})_{,j} + [a^{ij}(x,\epsilon u) - a^{ij}(x,0)]u_{,ij}$$

$$+ [(a^{ij})_{,j}(x,\epsilon u) - (a^{ij})_{,j}(x,0)]u_{,i} + \epsilon(a^{ij})_{,p}u_{,i}u_{,j}$$

$$= O(\epsilon)(|u| + |\nabla u|) \; , \tag{5.9}$$

where $w = u - h$. Setting $Lw = (A^{ij}w_{,i})_{,j}$, note that L is a uniformly elliptic operator in $\bar{D}$. To choose the domains D_α on which to obtain L^2 estimates for w, replace the conditions (2.3) and (2.4) on the auxilliary function f by

$$Lf \le 0 \; , \qquad |Lf| \le \delta^2 d \tag{5.10}$$

where δ and d are positive constants.

To find appropriate constraints under which one can derive L^2 bounds for w, we need to examine the L^2 integral of Lw. From (5.9),

$$(Lw)^2 \le \tau\epsilon^2[u^2 u,_{ij}u,_{ij} + u^2 u,_k u,_k + (a^{ij}),_p(a^{ij}),_p(u,_\ell u,_\ell)^2 + u^2 + u,_i u,_i] \tag{5.11}$$

for an explicit constant τ. One can estimate

$$\int_{D_\alpha}(Lw)^2 dx$$

in terms of data and volume integrals involving only u and its first derivatives by means of

<u>Lemma 7: Let $\alpha_1 \epsilon (0, 1)$. Then for $\alpha\ \epsilon(0, \alpha_1)$</u>

$$\begin{aligned}\int_{D_\alpha}(Lw)^2 dx \le k\epsilon^2\Bigg\{\int_\Sigma & u^2 |u,_i b^{jk}\left(n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k}\right)u,_j \\ & - (b^{jk}),_j u,_k u,_i n_i + (b^{jk}),_i u,_i u,_j n_k|ds + \int_{D_1}[u^2 \\ & + (1 + \epsilon|u|)^2(u^2 u,_i u,_i + (u,_j u,_j)^2) + (u,_i u,_i)(1 + u^2) \\ & + \epsilon^2 u^2(u^2 + u,_i u,_i) + a^{ij}_{,p}a^{ij}_{,p}(u,_i u,_i)^2 + \epsilon^2 u^2 u,_j u,_j)]dx\Bigg\}\end{aligned} \tag{5.12}$$

for an explicit constant k independent of ϵ and α.

Proof: Using the cutoff function ω of the last two sections and the ellipticity condition (5.3),

$$\int_{D_\alpha} u^2 u_{,ij} u_{,ij} dx \le \frac{1}{a_0} \int_{D_1} \omega^2 u^2 b^{jk} u_{,ij} u_{,ik} dx \quad . \tag{5.13}$$

Integration by parts gives

$$\begin{aligned} \int_{D_1} \omega^2 u^2 b^{jk} u_{,ij} u_{,ik} dx &= \int_{\Sigma_1} \omega^2 u^2 b^{jk} u_{,ij} u_{,i} n_k ds \\ &- 2 \int_{D_1} \omega\omega_{,k} u^2 b^{jk} u_{,ij} u_{,i} dx - 2 \int_{D_1} \omega^2 u u_{,k} b^{jk} u_{,ij} u_{,i} dx \\ &- \int_{D_1} \omega^2 u^2 (b^{jk} u_{,ij})_{,k} u_{,i} dx \quad . \end{aligned} \tag{5.14}$$

Rewrite the boundary term in (5.14) as

$$\begin{aligned} &\int_{\Sigma_1} \omega^2 u^2 b^{jk} u_{,ij} u_{,i} n_k ds \\ &= \int_{\Sigma_1} \omega^2 u^2 u_{,i} b^{jk} \left(n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k} \right) \cdot u_{,j} ds \end{aligned}$$

$$+\int_{\Sigma_1} \omega^2 u^2 u_{,i} n_i b^{jk} u_{,jk} ds$$

$$-\int_{\Sigma_1} \omega^2 u^2 u_{,i} b^{jk}\left(n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k}\right) u_{,j} ds$$

$$+\int_{\Sigma_1} \omega^2 u^2 u_{,i} n_i g \, ds - \int_{\Sigma_1} \omega^2 u^2 u_{,i} n_i (b^{jk})_{,j} u_{,k} ds \ . \qquad (5.15)$$

Since

$$n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k}$$

is a tangential derivative on Σ, the three terms on the right side of (5.15) involve only Cauchy data.

Examine now the volume integrals on the right side of (5.14). Using the A-G inequality and (5.3),

$$- 2\int_{D_1} \omega\omega_{,k} u^2 b^{jk} u_{,ij} u_{,i} dx - 2\int_{D_1} \omega^2 u u_{,k} b^{jk} u_{,ij} u_{,i} dx$$

$$\leq \delta \int_{D_1} \omega^2 u^2 b^{jk} u_{,ij} u_{,ik} dx +$$

$$K_1 \int_{D_1} b^{jk} b^{jk} [(u_{,i} u_{,i}) u^2 + (u_{,i} u_{,i})^2] dx \ , \qquad (5.16)$$

where δ is a positive constant which we may choose to be as small as we like and K_1 is a computable constant depending on δ. For the last integral on the right side of (5.14), note that

$$-\int_{D_1} \omega^2 u^2 (b^{jk} u_{,ij})_{,k} u_{,i} dx = -\int_{D_1} \omega^2 u^2 (b^{jk} u_{,j})_{,ki} u_{,i} dx$$

$$+ \int_{D_1} \omega^2 u^2 [(b^{jk})_{,i} u_{,j}]_{,k} u_{,i} dx \; . \tag{5.17}$$

The first integral on the right side of (5.17) can be written as

$$-\int_{D_1} \omega^2 u^2 (b^{jk} u_{,j})_{,ki} u_{,i} dx = -\int_{\Sigma_1} \omega^2 u^2 u_{,i} g(x,v,\text{grad } v) n_i ds$$

$$+ \int_{D_1} \omega^2 u^2 (\Delta u) g(x,v,\text{grad } v) dx$$

$$+ \int_{D_1} \left[(\omega^2 u^2)_{,i} u_{,i} \right] g(x,v,\text{grad } v) dx \tag{5.18}$$

where we have integrated by parts and used (5.1). Using the A-G inequality on the right side of (5.18),

$$- \int_{D_1} \omega^2 u^2 (b^{jk} u,_j),_{ki} u,_i dx \le - \int_{\Sigma_1} \omega^2 u^2 g(x,v,\text{grad } v) u,_i n_i ds$$

$$+ \delta \int_{D_1} \omega^2 u^2 b^{jk} u,_{ij} u,_{ik} dx + O(1) \int_{D_1} (u,_i u,_i)(u,_j u,_j + u^2) dx$$

$$+ O(\epsilon^2) \int_{D_1} u^2 (u^2 + u,_j u,_j) dx \ . \tag{5.19}$$

Integrating the second integral on the right side of (5.17) by parts,

$$\int_{D_1} \omega^2 u^2 [(b^{jk}),_i u,_j],_k u,_i dx = \int_{\Sigma_1} \omega^2 u^2 (b^{jk}),_i u,_i u,_j n_k ds$$

$$- \int_{D_1} \omega^2 u^2 (b^{jk}),_i u,_{ik} u,_j dx - \int_{D_1} (\omega^2),_k u^2 (b^{jk}),_i u,_i u,_j dx$$

$$- 2 \int_{D_1} \omega^2 u u,_i u,_j u,_k (b^{jk}),_i dx \ . \tag{5.20}$$

Using the A-G inequality and (5.3),

$$-\int_{D_1} \omega^2 u^2 (b^{jk})_{,i} u_{,ik} u_{,j} dx \le \delta \int_{D_1} \omega^2 u^2 b^{jk} u_{,ij} u_{,ik} dx$$

$$+ K_2 \int_{D_1} u^2 (b^{jk}_{,i} b^{jk}_{,i})(u_{,\ell} u_{,\ell}) dx \; . \qquad (5.21)$$

where K_2 is a computable constant depending on δ. Another use of the A-G inequality gives

$$-\int_{D_1} (\omega^2)_{,k} u^2 (b^{jk})_{,i} u_{,i} u_{,j} dx \le K_3 \int_{D_3} u^2 (b^{jk}_{,i} b^{jk}_{,i})(u_{,\ell} u_{,\ell}) dx$$

$$+ K_4 \int_{D_1} u^2 (u_{,i} u_{,i}) dx \; , \qquad (5.22)$$

for computable constants K_3 and K_4. Finally, use the A-G inequality to bound the last integral in (5.20) by

$$K_5 \int_{D_1} u^2 (b^{jk}_{,i} b^{jk}_{,i})(u_{,\ell} u_{,\ell}) dx + K_6 \int_{D_1} (u_{,i} u_{,i})^2 dx \; , \qquad (5.23)$$

for computable constants K_5 and K_6.

Combining (5.13) through (5.23), we find that for $0 < \alpha < \alpha_1$,

$$\int_{D_\alpha} u^2 u_{,ij} u_{,ij} dx \le K_7 \epsilon^2 \left\{ \int_{\Sigma_1} w^2 u^2 \left[u_{,i} b^{jk} \left(n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k} \right) u_{,j} \right.\right.$$

$$\left. - (b^{jk})_{,j} u_{,k} u_{,i} n_i + (b^{jk})_{,i} u_{,i} u_{,j} n_k \right] ds$$

$$+ \int_{D_1} [(1 + \epsilon|u|)^2 (u^2 u_{,i} u_{,i} + (u_{,i} u_{,i})^2) + \epsilon^2 u^2 (u^2 + u_{,i} u_{,i})$$

$$\left. + \epsilon^2 (a^{jk}_{,p} a^{jk}_{,p}) u^2 u_{,\ell} u_{,\ell}] dx \right\}$$

for an explicit constant K_7. This, combined with (5.11), gives the result of Lemma 7.

Now impose the constraint that the term in braces in the statement of Lemma 7 is $O(\epsilon^{-q})$ for some $q < 4$. Thus,

$$\int_{D_\alpha} (Lw)^2 dx = O(\epsilon^{2-q}) \quad . \tag{5.24}$$

We proceed to derive a differential inequality for the functional

$$F(\alpha) = Q + \int_0^{\alpha} (\alpha-\eta)\left\{ \int_{D_\eta} [wLw + A^{ij}w,_i w,_j]dx \right\} d\eta \qquad (5.25)$$

where Q is given by

$$Q = k_0 \int_\Sigma w^2 ds + k_1 \int_\Sigma w,_i w,_i ds + k_2 \epsilon^{2-q} \ . \qquad (5.26)$$

As in section 3, one can choose the constants k_i in (5.26) so that an analog of (3.17) holds, i.e.,

$$\frac{1}{2}\left[\int_{D_\alpha} rw^2 dx + Q \right] \le F(\alpha) \le \frac{d+1}{2} \left[\int_{D_\alpha} rw^2 dx + Q \right] \qquad (5.27)$$

where $r = A^{ij}f,_i f,_j$. Assuming that

$$\int_\Sigma (w^2 + w,_i w,_i) ds = 0(\epsilon^{2-q}) \ , \qquad (5.28)$$

we are prepared to state the remaining estimates, which are understood to hold on the interval $(0, \alpha_1)$. Here, the proofs are very similar to those found in Payne (14).

Lemma 8: If $F(\alpha)$ is given by (5.25), then

$$|F'| \leq F' + K_1 F \tag{5.29}$$

for a computable constant K_1.

Lemma 9: If $F(\alpha)$ is given by (5.25), then

$$\int_{D_\alpha} A^{ij} w_{,i} w_{,j} dx - 2 \int_{D_\alpha} r^{-1} [A^{ij} w_{,i} f_{,j}]^2 dx \geq - K_2 F' - K_3 F$$

for computable constants K_2 and K_3.

One may now conclude as in section 2.1, that on the interval $(0, \alpha_1)$,

$$FF'' - (F')^2 \geq - C_1 FF' - C_2 F^2$$

for explicit constants C_1 and C_2. Assuming that

$$\int_{D_1} u^2 dx = 0(\epsilon^{-2}) , \qquad \int_{D_1} h^2 dx = 0(\epsilon^{-2}) , \tag{5.30}$$

we have

Theorem 3: If u and h are solutions of (5.7) and (5.8), respectively, which satisfy (5.24), (5.28), and (5.30), then for $0 < \alpha , \alpha_1$

$$\int_{D_\alpha} (v - \epsilon h)^2 dx = O(\epsilon^{\gamma(\alpha)})$$

with $\gamma(\alpha) = (4 - q)(\sigma - \sigma_1)/(1 - \sigma_1)$, $\sigma = \exp(-C_1\alpha)$, and $\sigma_1 = \exp(-C_1\alpha_1)$.

As in the other sections of this chapter, one can find a continuous dependence inequality for the Dirichlet integral of $v - \epsilon h$. Introduce the function μ as in section 3 with $\beta < \alpha_2$ and α_3 a fixed number between α_2 and α_1. Then

$$a_0 \int_{D_\beta} (u-h)_{,i}(u-h)_{,i} dx \le \int_{D_{\alpha_3}} \mu^2 A^{ij} w_{,i} w_{,j} dx$$

$$= \int_{\Sigma_{\alpha_3}} \mu^2 A^{ij} w_{,i} w n_j ds - \int_{D_{\alpha_3}} \mu^2 wLwdx$$

$$- \int_{D_{\alpha_3}} (\mu^2)_{,j} A^{ij} ww_{,i} dx = \int_{\Sigma_{\alpha_3}} [\mu^2 A^{ij} ww_{,i} n_j$$

$$- \frac{1}{2}(\mu^2)_{,j} A^{ij} w^2 n_i] ds - \int_{D_{\alpha_3}} \mu^2 wLwdx$$

$$+ \frac{1}{2} \int_{D_{\alpha_3}} w^2 [A^{ij}(\mu^2)_{,j}]_{,i} dx \, . \tag{5.31}$$

Use the A-G inequality and the estimates of this section to conclude that

$$\int_{D_\beta} (v - \epsilon h)_{,i}(v - \epsilon h)_{,i}dx = o\left(\epsilon^{\gamma(\alpha_3)}\right) .$$

6. Concluding Remarks

The arguments in this work do not yield continuous dependence results if $p = 6$ in the constraints (3.25) and (4.19), or $q = 4$ in (5.24). Such constraints would be desirable since they would not impose any apriori "smallness" conditions on volume integrals of the solutions or their derivatives. It is not clear, however, that continuous dependence results could be obtained from such constraints.

Acknowledgement--The author wishes to thank Professor L.E. Payne for his assistance in the preparation of this work and the thesis from which it is derived.

REFERENCES

[1] L. E. Adelson, Singular perturbation of improperly posed problems, SIAM J. Math. Anal., 4 (1973), pp. 344-366.

[2] L. E. Adelson, Singular perturbation of an improperly posed Cauchy problem, SIAM J. Math. Anal., 5(1974), pp. 417-424

[3] K. Ames, Comparison results for related properly and improperly posed Cauchy problems for second-order operator equations, J. Diff. Eq., 44 (1982), pp. 383-399

[4] K. Ames, On the comparison of solutions of related properly and improperly posed Cauchy problems for first-order operator equations, SIAM J. Math. Anal., 13 (1982), pp. 594-606.

[5] A. Bennett, Continuous dependence on modeling in the Cauchy problem for second-order nonlinear partial differential equations, Doctoral thesis, Cornell University, Ithaca, N.Y., 1986.

[6] J. Conlan and G. N. Trytten, Pointwise bounds in the Cauchy problem for elliptic systems of partial differential equations, Arch. Rational Mech. Anal., 22 (1966), pp. 143-152.

[7] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, Berlin, 1983

[8] J. Hadamard, Lecures on Cauchy's problem in Linear Partial Differential Equations, Yale University Press, New Haven, 1983

[9] F. John, Numerical solution of the heat equation for preceding time, Ann. Mat. Pure Appl., 40 (1955), pp. 129-142.

[10] F. John, Continuous dependence on the data for solutions of partial differential equations with a prescribed bound, Comm. Pure Appl. Math., 13 (1960), pp. 551-585

[11] M. M. Lanrentiev, On the Cauchy problem for the Laplace equation. Ivest. Akad. Nauk. SSSR., 20 (1956), pp. 819-842

[12] H. Levine, Logarithmic convexity and the Cauchy problem for some abstract second order differential inequalities, J. Diff. Eq., 8 (1970), pp. 34-55.

[13] L. E. Payne, Bounds in the Cauchy problem for the Laplace equation. Arch. Rational Mech. Anal., 5 (1960), pp. 35-45.

[14] L. E. Payne, On a priori bounds in the Cauchy problem for elliptic equations, SIAM J. Math. Anal., 1 (1970), pp. 82-89

[15] L. E. Payne, Improperly Posed Problems in Partial Differential Equations, Regional Conference Series in Applied Mathematics, 22 (1974).

[16] C. Pucci, Sur problemi di Cauchy non "ben poste." Rend. Accad. Naz. Lincei, 18 (1955), pp. 473-477.

[17] C. Pucci, Discussione del problema di Cauchy per le equazione di tipo ellitico, Ann. Mat. Pura Appl., 46 (1958), pp. 131-153

[18] P. W. Schaefer, On the Cauchy problem for an elliptic system, Arch. Rational Mech. Anal., 20 (1965), pp. 660-673

[19] P. W. Schaefer, On uniqueness, stability and pointwise estimates in the Cauchy problem for coupled elliptic systems, Quart. Appl. Math., 31 (1973), pp. 321-328.

[20] P. W. Schaefer, A priori bounds in the Cauchy problem for coupled elliptic systems, Applicable Analysis, 4 (1974), pp. 207-221.

[21] G. N. Toytten, Pointwise bounds for solutions in the Cauchy problem for elliptic equations, Arch. Rational Mech. Anal., 13 (1963), pp. 222-244.

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